STANDARD 15 — CONCEPTUAL BUILDING BLOCKS OF CALCULUS

K-12 Overview

> All students will develop an understanding of the conceptual building blocks of calculus and will use them to model and analyze natural phenomena.

Descriptive Statement

The conceptual building blocks of calculus are important for everyone to understand. How quantities such as world population change, how fast they change, and what will happen if they keep changing at the same rate are questions that can be discussed by elementary school students. Another important topic for all mathematics students is the concept of infinity — what happens as numbers get larger and larger and what happens as patterns are continued indefinitely. Early explorations in these areas can broaden students' interest and understanding of an important area of applied mathematics.

Meaning and Importance

Calculus is the mathematics used to describe processes evolving in space or time. **How quantities change** — the velocity of a car as its position changes over time or the area of a square as its sides lengthen — and **what happens in the long run** are central themes of mathematics and its application to the real world. Calculus is used to describe an exact result as the **limit of a sequence of approximations.** Calculus is essential for understanding the physical world and indispensable in economics and industry. Engineers and physicists use calculus to calculate motion in response to forces. Businessmen and economists use calculus to find optimal solutions to pricing and production. An intuitive feel for the mathematics of infinity, limit, and change are accessible and necessary for all students.

Although some students will go on to study these concepts in a formal calculus course, this standard does **not** advocate the formal study of calculus in high school for all students or even for all college-intending students. Rather, it calls for providing opportunities for all students to informally investigate the central ideas of calculus. Considering these concepts will contribute to a deeper understanding of the function concept and its usefulness in representing and answering questions about real-world situations.

K-12 Development and Emphases

Students at the elementary level should understand the concept of **linear growth** (constant increments). For instance, a savings account grows linearly if equal deposits are made at regular intervals and the account bears no interest. This idea and its extensions can be introduced and mastered without using the mathematical formalism of functions, which is introduced later in the middle grades. Beginning in elementary school, children can accumulate records of processes which exhibit linear growth. Some examples are mileage accumulated by traveling between home and school, cumulative expenses for school lunches, and cumulative volume of cereal consumed if everyone in the class eats a bowl every morning. Children should learn to recognize linear growth and compare it to the more irregular pattern of increases in their own height, the height of a bean plant, cumulative rainfall, class consumption of paper, and real expenditures. Children in elementary school can be introduced to **exponential growth** (ever-increasing increments) through the discovery that if every pair of rabbits produces two rabbits each month (one new rabbit for every existing rabbit) then in less than two years there would be more than a million rabbits!

Middle school students should be moving beyond the concrete and pictorial representations used in the elementary grades to more symbolic ones, involving functions and equations. They should use graphing calculators and computers to develop and analyze graphical representations of the changes represented in the tables, and to produce linear and quadratic regression models of the data. They should apply their knowledge of decimals to solving problems involving compound interest, making use of a calculator to determine, for example, the yield of a given investment or the length of time it would take for an investment to double. In high school, students can apply their knowledge of exponents, algebra, and functions to solve these and other more difficult problems, with applications to growth in economics and biology (e.g. population explosion), algebraically and graphically.

Throughout their school years, students should be examining a variety of situations where populations and other quantities **change over time**, and use the mathematical tools at their disposal to describe and analyze this change. As they progress, the situations considered should become more complex; students who experiment with constant motion in their early years will be able to understand the motion of projectiles (a ball thrown into the air, for example) by the time they complete high school.

Similarly, students should be aware how changing the linear dimensions of an object — such as its height, length, or diameter — affects its area and volume. In the early years children should be involved in hands-on experiments which illustrate this; for example, they might find that doubling the diameter of a circular can (of fixed height) increases the volume four-fold by filling the smaller can with water or rice and emptying it into the larger one. By the time students are familiar with variables, this intuition will provide them with the information they need to understand formulas such as those involving volume.

In many settings, the kind of change that takes place over time is repetitive and cumulative, and an important question that should be discussed is what happens in the long run. The principal tool for understanding and discussing such questions is the concept of infinite sequences and the types of patterns that emerge from them. Thus a second central theme is that of **infinity.**

Students are fascinated with the mysteries of large numbers and “infinity,” and that excitement should be nourished and be used, as with other “teachable moments,” to motivate the learning of more mathematics. Primary students enjoy naming their “largest” number or proudly declaring that there is no largest! In the early years, large numbers and their significance should be discussed, as should the idea that one can extend simple processes forever (e.g., keep adding 2, keep multiplying by 3).

Once students have familiarity with fractions and decimals, these notions can be extended. *What happens when you keep dividing by 2? By 10? Can you find a fraction between 0.999 and 1? What decimal comes just before 1?* Students should explore and experiment with infinitely repeating decimals and other infinite series, where they can make tables and look for patterns. They should learn that by repeated iteration of simple processes you can get better and better answers in both arithmetic (with increased decimal accuracy) and in geometry (with more accurate estimates of the area and volume of irregular objects).

Although the concept of a **limiting value** (or a limit) may appear inaccessible to K-8 students, this basic notion of calculus can be explored through the process of measuring the area of a region. Students can be provided with diagrams of a large circular (or irregular) region, say a foot in diameter, and a large supply of tiles of different square sizes. By covering the space inside the region (with no protrusions!) with 4" tiles, then with 2" tiles, then with 1" tiles, then with .5" tiles, students can gain an appreciation that the smaller the unit used, the larger the area obtained. They will recognize that the space cannot be filled completely with small tiles, yet, at the same time, the sum of the areas of the smaller tiles gets closer and closer to the actual area of the region.

IN SUMMARY, these kinds of experiences will provide a good foundation for the notions of limits, infinity, and changes in quantities over time. Such concepts find many applications in both science and mathematics, and students will feel much more comfortable with them if they begin to develop these concepts in the early grades.

NOTE: *Although each content standard is discussed in a separate chapter, it is not the intention that each be treated separately in the classroom. Indeed, as noted in the Introduction to this Framework, an effective curriculum is one that successfully integrates these areas to present students with rich and meaningful cross-strand experiences.*

Standard 15 — Conceptual Building Blocks of Calculus — Grades K-2

Overview

Students in the early primary grades bring to the classroom intuitive notions of the meaning of such terms as *biggest, largest, change*, and so forth. While they may not know the names of large numbers, they certainly have a sense of "largeness." The cumulative process indicators related to this standard for grades K-2 deal primarily with investigating patterns of growth and change over time.

Students in grades K-2 should investigate many different types of patterns. For some of these patterns, such as 2, 4, 6, 8, ... , the same number is added (or subtracted) to each number to get the next number in the sequence. When these patterns are represented with a bar graph, the tops of the bars can be connected by a straight line, so the pattern represents **linear growth.** Older students should also see patterns that grow more rapidly, such as 2, 4, 8, ... These growing patterns involve **exponential growth**; each number in the series is multiplied (or divided) by the same number to get the next one. In this situation, when the tops of the bars on a graph are connected, they do not form a straight line. These types of patterns can be investigated very easily by using calculators to do the computation; students enjoy making the numbers bigger and bigger by using a constant addend (e.g., *2* + *2* = = =) or a constant multiplier (e.g., *2* x *2* = = =). (Note that some calculators require different keystrokes to achieve this effect.) By relating these problems to concrete situations, such as the growth of a plant, students begin to develop a sense of **change over time**.

Students also begin to develop a sense of change with respect to **measurement**. Students begin to measure the length of objects by using informal units such as paperclips or Unifix cubes; they should note that it takes more small objects to measure a given length than large ones. By the end of second grade, they begin to describe the area of objects by counting the number of squares that cover a figure. Again, they should note that it takes more small squares to cover an object than it does large ones. They should also begin to investigate what happens to the area of a square when each side is doubled. Students also need to develop volume concepts by filling containers of different sizes. They might use two circular cans, one of which is twice as high and twice as wide as the other, to find that the large one holds eight times as much as the small one. Measurement may also lead to the beginnings of the idea of **a limiting value** for young children. For example, the size of a dinosaur footprint might be measured by covering it with base ten blocks. If only the 100 blocks are used, then one estimate of the size of the footprint is found; if unit blocks are used, a more precise estimate of the size of the footprint can be found.

Students in grades K-2 should also begin to look at concepts involving **infinity**. As they learn to count to higher numbers, they begin to understand that, no matter how high they count, there is always a bigger number. By using calculators, they can also begin to see that they can continue to add two to a number forever and the result will just keep getting bigger.

The conceptual underpinnings of calculus for students in grades K-2 are closely tied to their developing understanding of number sense, measurement, and pattern. Additional activities relating to this standard can be found in the chapters discussing these other standards.

Standard 15 — Conceptual Building Blocks of Calculus — Grades K-2

Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in kindergarten and grades 1 and 2.

Experiences will be such that all students in grades K-2:

1. **Investigate and describe patterns that continue indefinitely.**
    - Students model repeating patterns with counters or pennies. For example, they repeatedly add two pennies to their collection and describe the results.
    - Students create repeating patterns with the calculator. They enter any number such as 10, and then add *1* for *10 + 1 = = =* . . . . The calculator will automatically repeat the function and display 11, 12, 13, 14, etc. each time the = key is pressed. (Some calculators may need to have the pattern entered twice: *10 + 1 = + 1 = = =* . . . etc. Others may use a key sequence such as *1++10 = = =*. . . .) Students may repeatedly add (or subtract) any number.
    - Second graders create a pattern with color tiles. They start with one square and then make a larger square that is two squares long on each side; they note that they need four tiles to do this. Then they make a square that is three squares long on each side; they need nine tiles to do this. They make a table of their results and describe the pattern they have found.
    - Students investigate a doubling (growing) pattern with Unifix cubes. They begin with one cube and then “win” another cube. Then they have two cubes and “win” two more. They continue this pattern, each time “winning” as many cubes as they already have. Repeating this process, they begin to see how quickly the number of cubes grows. They investigate this further using a calculator.
    - Students start with a rectangular sheet of paper that represents a cake. They simulate eating half of the cake by cutting the sheet in half and removing one of the halves. They eat half of what is left and continue this process. They describe the pattern, noting that after they repeat this about ten times, the cake is essentially gone.

2. **Investigate and describe how certain quantities change over time.**
    - Students keep a daily record of the temperature both inside and outside the classroom. They graph these temperatures and look at the patterns.
    - Students keep a monthly record of their height and record the data collected on a bulletin board. At the end of the school year, they describe what happened over time.
    - Students play catch with a ball in the school playground. One person counts out the number of times the ball is thrown, the other counts out the distance that it travels, a third person adds that distance to the total, and a fourth person records the totals. Afterwards they discuss how the total distance changes over time; they recognize that the same amount is added repeatedly.
    - Students study the changes in the direction and length of the shadow of a paper groundhog at

different times of the day. They relate these observations to the position of the sun (e.g., as the sun gets higher, the shadow gets shorter).

- Students discuss how ice changes to water as it gets hotter. They talk about how it snows in January or February but rains in April or May.
- Students plant seeds and watch them grow. They write about what they see and measure the height of their plants at regular time intervals. They discuss how changes in time result in changes in the height of the plant. They also talk about how other factors might affect the growth of the plant, such as light and water.

3. Experiment with approximating length, area, and volume, using informal measurement instruments.

- Students measure the width of a bookcase using the 10-rods from a base ten blocks set. They record this length (perhaps as 6 rods or 60 units). Then they measure the bookcase using ones cubes; some of the students decide that it is easier just to add some ones cubes to the 10-rods that they have already used. They find that the bookcase is actually closer to 66 units long. They decide that they can get a better estimate of length when they use smaller units.
- Students use pattern blocks to cover a picture of a turtle. They count how many of each type of block (green triangle, yellow hexagon, etc.) they used. They make a bar graph that shows how many blocks each student used. They discuss why some students used more blocks than others and what they could do to increase or decrease the number of blocks used.
- Students play with containers of various sizes, transferring water from one container to another. They note that it takes two cups of water to fill a small milk carton. A pitcher holds three milk cartons of water, but four milk cartons overflow the pitcher. Then they find that it takes seven cups to fill the pitcher even though three milk cartons is only six cups. They decide that the smaller container gives a better idea of how much the pitcher will hold.
- Students find the area of huge dinosaur footprints that they find taped to the classroom floor. They first try to fit as many green 4" tiles as possible into a footprint without any overlapping, and without any tiles sticking out of the footprint. Before removing the green tiles, they cover them with blue 2" tiles, and count the number of blue 2" tiles used. Then they remove the green tiles and try to fit more blue 2" tiles into the footprint without overlapping; they discover that they can fit more and discuss why that is the case. They repeat this, using red 1" tiles. They notice that with smaller tiles, less of the footprint is uncovered, so that the smaller tiles provide a better estimate of the footprint's size.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards.*

Standard 15 — Conceptual Building Blocks of Calculus — Grades 3-4

Overview

Students in grades 3 and 4 continue to develop the conceptual building blocks of calculus primarily through their work with patterns and changes over time. Students investigate a variety of patterns, using physical materials and calculators as well as pictures. In some of the patterns investigated, a constant is added to, or subtracted from, each number to get the next number in the sequence. These patterns, involving repeated operations, show **linear growth**; when these patterns are represented with a bar graph, the tops of the bars can be connected by a straight line. Examples of such patterns include skip counting, starting the week with $5 and paying 75¢ each day for lunch, or enumerating the multiples of 9. Other patterns should involve multiplying or dividing a number by a constant to get the next number in the sequence. These growing patterns illustrate **exponential growth**, as is the pattern which results when you start with two guppies (one male and one female) and the number of guppies doubles each week. Patterns should also include looking at **changes over time**, since these types of patterns are extremely important not only in mathematics but also in science and social studies. Students might chart the height of plants over time, the number of teeth lost each month throughout the school year, or the temperature outside the classroom over the course of several months.

Students continue to develop their understanding of **measurement**, gaining a greater understanding of the approximate nature of measurement. Students can guess at the length of a stick that is between 3 and 4 inches long, saying it is about 3½ inches long and recognizing when this is a better approximation than either 3 or 4. They can use grids of different sizes to approximate the area of a puddle, recognizing that the smaller the grid the more accurate the measurement. They can begin to consider how one might measure the amount of water in a puddle, coming up with alternative strategies and comparing them to see which would be more accurate. As they develop a better understanding of volume, they may use cubes to build a solid, build a second solid whose sides are all twice as long as the first, and then compare the number of cubes used to build each solid. The students may be surprised to find that it takes eight times as many cubes to build the larger solid!

Students continue to develop their understanding of **infinity** in grades 3 and 4. Additional work with counting sequences, skip counting, and calculators further reinforces the notion that there is always a bigger number. Taking half of something (like a rectangular cake or a sheet of paper) repeatedly suggests that there is always a number that is still closer to zero.

As students develop the conceptual underpinnings of calculus in third and fourth grades, they are also working to develop their understanding of numbers, patterns, measurement, data analysis, and mathematical connections. Additional ideas for activities relating to this standard can be found in the chapters discussing these other standards.

Standard 15 — Conceptual Building Blocks of Calculus — Grades 3-4

Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 3 and 4.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 3-4 will be such that all students:

1. Investigate and describe patterns that continue indefinitely.

- Using the constant multiplier feature of a calculator, students see how many times 1 must be doubled before one million is reached. They might first guess the number of steps to one million, and to half a million.
- Students start with a long piece of string. They fold it in half and cut it in two, setting aside one piece. Then they take the remaining half, fold it in half, cut it apart, and set aside half. They continue this process. They discuss how the length of the string keeps getting smaller, half as much each time, so that after about ten cuts, there is essentially no string left. Some students may understand that the process could keep going for several more steps, if we could only cut more carefully, and some may realize that in theory the process could continue forever.
- Students count out 1, 2, 3, 4, 5, 6, ... , and recognize that this pattern could continue forever. They also count out other patterns, like the even numbers, or the square numbers, or skip-counting by 3s starting with 2, and recognize that these patterns also could be continued indefinitely.
- Students investigate the growth patterns of sunflowers, pinecones, pineapples, or snails to study the natural occurrence of spirals.

2. Investigate and describe how certain quantities change over time.

- Students begin with a number like 5, add 3 to it, add 3 to it again, and repeat this five times. They record the results in a table and make a bar graph which represents the numbers that they have generated. They draw a straight line connecting the tops of the bars. They experiment with numbers other than 5 or 3 to see if the same thing occurs.
- Students begin with a number less than 10, double it, and repeat the doubling at least five times. They record the results of each doubling in a table and make a bar graph which represents the numbers they have generated. They discuss whether they can connect the tops of the bars with a straight line.
- Students measure the temperature of a cup of water with ice cubes in it every fifteen minutes over the course of a day. They record their results (time passed and temperature) in a table and plot this information on a coordinate grid to make a line graph. They discuss how the temperature changes over time and why. Initially the temperature will increase rapidly, but as the water warms up, its temperature will increase more slowly until it essentially reaches

room temperature.

- Students are given several examples of bar graphs with straight lines connecting the tops of the bars. They are asked to describe a motion scenario which reflects the data. For example, they might indicate that a graph reflects their running to an after-school activity, staying there for an hour, and then slowly walking home to do their chores.
- Students keep a monthly record of their height and record the data collected on a bulletin board. At the end of the school year, they describe what happened over time. They also find each month the average height of all the students in the class, and discuss how the change in average height over the year is similar to, and how it is different from, the change in height over the year of the individual students in the class.
- Students plant some seeds in vermiculite and some in soil. They observe the plants as they grow, measuring their height each week and recording their data in tables. They examine not only how the height of each plant changes as time passes but also whether the seeds in vermiculite or soil grow faster.

3. Experiment with approximating length, area, and volume, using informal measurement instruments.

- Students measure the length of their classroom using their paces and compare their results. They discuss what would happen if the teacher measured the room with her pace.
- Students use pattern blocks to cover a drawing of a dinosaur with as few blocks as they can. They record the number of blocks of each type used in a table and then discuss their results, making a frequency chart or bar graph of the total number of blocks used by each pair of students. Then they try to cover the same drawing with as many blocks as they can. They again record and discuss their results and make a graph. They look for connections between the numbers and types of blocks used each time. Some students simply trade blocks (e.g., a hexagon is traded for six triangles), while other students try to use all tan parallelograms since that seems to be the smallest block. (It actually has the same area as the triangle, however.)
- Students compare the volumes of a half-gallon milk carton, a quart milk carton, a pint milk carton, and a half-pint milk carton. They also measure the length of the side of the square base of each carton and its height. They make a table of their results and look for patterns. The students notice that the difference between the height measurements is not the same as the difference between the volume. The differences in volume grow more quickly than the differences in the heights. They see how many small cubes or marbles fill up each of their containers, and they try to explain why more than twice as many fit into a quart container than a pint container.

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Standard 15 — Conceptual Building Blocks of Calculus — Grades 5-6

Overview

Students in grades 5 and 6 extend and clarify their understanding of patterns, measurement, data analysis, number sense, and algebra as they further develop the conceptual building blocks of calculus. Many of the basic ideas of calculus can be examined in a very concrete and intuitive way in the middle grades.

Students in grades 5 and 6 should begin to distinguish between patterns involving **linear growth** (where a constant is added or subtracted to each number to get the next one) and **exponential growth** (where each term is multiplied or divided by the same number each time to get the next number). Students should recognize that linear growth patterns change at a constant rate. For example, a plant may grow one inch every day. They should also begin to see that if these patterns are graphed, then the graph looks like a straight line. They may model this line by using a piece of spaghetti and use their graph to make predictions and answer questions about points that are not included in their data tables. In contrast, exponential growth patterns change at an increasingly rapid rate; if you start with one penny and double that amount each day, you receive more and more pennies each day as time goes on. Students should note that the graphs of these situations are not straight lines. At this grade level, students should also begin to imagine processes that could in theory continue forever even though they could not be carried out in practice; for example, although in practice a cake can be repeatedly divided in half only about ten times, nevertheless it is possible to imagine continuing to divide it into smaller and smaller pieces.

Many of the examples used should come from other subject areas, such as science and social studies. Students might look at such linear relationships as profit as a function of selling price, but they should also consider nonlinear relationships such as the amount of rainfall over time. Students should look at functions which have “holes” or jumps in their graphs. For example, if students make a table of the parking fees paid for various amounts of time and then plot the results, they will find that they cannot just connect the points; instead there are jumps in the graph where the parking fee goes up. A similar situation exists for graphs of the price of a postage stamp or the minimum wage over the course of the years. Many of the situations investigated by students should involve such **changes over time**. Students might, for example, consider the speed of a fly on a spinning disk; as the fly moves away from the center of the disk, he spins faster and faster. Students might be asked to write a short narrative about the fly on the disk and draw a graph of the fly’s speed over time that matches their story.

As students begin to explore the decimal equivalents for fractions, they encounter non-terminating decimals for the first time. Students should recognize that calculators often use approximations for fractions such as .33 for 1/3. They should look for patterns involving decimal representations of fractions, such as recognizing which fractions have terminating decimal equivalents and which do not. Students should take care to note that π is a nonterminating, nonrepeating decimal; it is not exactly equal to 22/7 or 3.14, but these approximations are fairly close to the actual value of π and can usually be used for computational purposes. The examination of decimals extends students’ understanding of **infinity** to very small numbers.

Students in grades 5 and 6 continue to develop a better understanding of the approximate nature of **measurement**. Students are able to measure objects with increasing degrees of accuracy and begin to consider significant digits by looking at the range of possible values that might result from computations with approximate measures. For example, if the length of a rectangle to the nearest centimeter is 10 cm and its width to the nearest centimeter is 5 cm, then the area is about 50 square centimeters. However, the rectangle might really be as small as 9.5 cm x 4.5 cm, in which case the area would only be 42.75 square centimeters, or it might be almost as large as 10.5 cm x 5.5 cm, with an area of 57.75 square centimeters. Students should continue to explore how to determine the surface area of irregular figures; they might, for example, be asked to develop a strategy for finding the area of their hand or foot. They should do similar activities involving volume, perhaps looking for the volume of air in a car. Most of their work in this area in fifth and sixth grade will involve using squares or cubes to approximate these areas or volumes.

Standard 15 — Conceptual Building Blocks of Calculus — Grades 5-6

Indicators and Activities

The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 5 and 6.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 5-6 will be such that all students:

4. Recognize and express the difference between linear and exponential growth.

- Students develop a table showing the sales tax paid on different amounts of purchases, graph their results, note that the graph is a straight line, and recognize that this situation represents a constant rate of change, or linear growth.
- Students make a table showing how much money they would have at the end of each of eight years if $100 was invested at the beginning and the investment grew by 10% each year. They note that the graph of their data is not a straight line; this graph represents exponential growth.
- Students make a table showing the value of a car as it depreciates over time. They note that the graph of their data is not a straight line; this graph represents "exponential decay."
- Students are presented stories which represent real life occurrences of linear and exponential growth and decay over time, and are asked to construct graphs which represent the situation and indicate whether the change is linear, exponential, or neither.

5. Develop an understanding of infinite sequences that arise in natural situations.

- Students make equilateral triangles of different sizes out of small equilateral triangles and record the number of small triangles used for each larger triangle. These numbers are called *triangular numbers*. If the following triangular pattern is continued indefinitely, then the number of 1s in the first row, the number of 1s in the first two rows, the number of 1s in the first three rows, etc. form the sequence of triangular numbers. The triangular numbers also emerge from the handshake problem: *If each two people in a room shake hands exactly once, how many handshakes take place altogether?* If the answers are listed for 2, 3, 4, 5, 6, 7, ... people, the numbers are again the triangular numbers 1, 3, 6, 10, 15, 21, ...

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          1
        1   1
      1   1   1
    1   1   1   1
  1   1   1   1   1
1   1   1   1   1   1
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- Students imagine cutting a sheet of paper into half, cutting the two pieces into half, cutting the four pieces into half, and continuing this over and over again, for about 25 times. Then they imagine taking all of the little pieces of paper and stacking them on top of one another. Finally, they estimate how tall that stack would be.

- Students describe, analyze, and extend the Fibonacci sequence (1, 1, 2, 3, 5, 8, ...). They research occurrences of this sequence in nature, such as sunflower seeds, the fruit of the pineapple, and the rabbit problem. They create their own Fibonacci-like sequences, using different starting numbers.

6. Investigate, represent, and use non-terminating decimals.

- Students use their calculators to find the decimal equivalent for 2/3 by dividing 2 by 3. Some of the students get an answer of 0.66667, while others get 0.6666667. They do the problem by hand to try to understand what is happening. They decide that different calculators round off the answer after different numbers of decimal places. The teacher explains that the decimal for 2/3 can be written exactly as $.\overline{6}$.

- Students have been looking for the number of different squares that can be made on a 5 x 5 geoboard and have come up with 1x1, 2x2, 3x3, 4x4, and 5x5 squares. One student finds a different square, however, whose area is 2 square units. The students wonder how long the side of the square is. Since they know that the area is the length of the side times itself, they try out different numbers, multiplying *1.4 x 1.4* on their calculators to get 1.89 and *1.5 x 1.5* to get *2.25*. They keep adding decimal places, trying to get the exact answer of 2, but find that they cannot, no matter how many places they try!

7. Represent, analyze, and predict relations between quantities, especially quantities changing over time.

- Students study which is the better way to cool down a soda, adding lots of ice at the beginning or adding one cube at a time at one minute intervals. Each student first makes a prediction and the class summarizes the predictions. Then the class collects the data, using probes and graphing calculators or computers and displays the results in table and graph form on the overhead. The students compare the graphs and write their conclusions in their math notebooks. They discuss the reasons for their results in science class.

- Students make a graph that shows the price of mailing a letter from 1850 through 1995. Some of the students begin by simply plotting points and connecting them but soon realize that the price of a stamp is constant for a period of time and then abruptly jumps up. They decide that parts of this graph are like horizontal lines. The teacher tells them that mathematicians call this a “step function”; another name for this kind of graph is a piecewise linear graph because the graph consists of linear pieces.

- Students review Mark’s trip home from school on his bike. Mark spent the first few minutes after school getting his books and talking with friends, and left the school grounds about five minutes after school was over. He raced with Ted to Ted’s house and stopped for ten minutes to talk about their math project. Then he went straight home. The students draw a graph showing the distance covered by Mark with respect to time. Then, with the teacher’s help, the class constructs a graph showing the speed at which Mark traveled with respect to time. The students then write their own stories and generate graphs of distance vs. time and graphs of speed vs. time.

8. **Approximate quantities with increasing degrees of accuracy.**

- Students find the volume of a cookie jar by first using Multilink cubes (which are 2 cm on a side) and then by using centimeter cubes. They realize that the second measurement is more accurate than the first.
- Students measure the circumference and diameter of a paper plate to the nearest inch and then divide the circumference by the diameter. They repeat this process, using more accurate measures each time (to the nearest half-inch, to the nearest quarter-inch, etc.). They see that the quotients get closer and closer to π.
- Using a ruler, students draw an irregularly shaped pentagon on square-grid paper, taking care to locate the vertices of the pentagon at grid points. They estimate the area of the pentagon by counting the number of squares completely inside the pentagon and adding to it an estimate of the number of full square that the partial squares inside the pentagon would add up to. Then they divide the pentagon into triangles and rectangles and find the area of the pentagon as a sum of the areas of the triangles and rectangles. They compare the results and explain any difference.

9. **Understand and use the concept of significant digits.**

- Students measure the length and width of a rectangle in centimeters and find its area. Then they measure its length and width in millimeters and find the area. They note the difference between these two results and discuss the reasons for such a difference. Some of the students think that, since the original measurements were correct to the nearest centimeter, then the result would be correct to the nearest square centimeter, while the second measurements would be correct to the nearest square millimeter. However, when they experiment with different rectangles, for example, one whose dimensions are 3.2 by 5.2 centimeters, they find that the area of 15 square centimeters is not correct to the nearest centimeter.
- Students find the area of a "blob" using a square grid. First, they count the number of squares that fit entirely within the blob (no parts hanging outside). They say that this is the least that the area could be. Then they count the number of squares that have any part of the blob in them. They say that this is the most that the area could be. They note that the actual area is somewhere between these two numbers.

10. **Develop informal ways of approximating the surface area and volume of familiar objects, and discuss whether the approximations make sense.**

- Students trace around their hand on graph paper and count squares to find an approximate value of the area of their hand. They use graph paper with smaller squares to find a better approximation.
- Students work in groups to find the surface area of a leaf. They describe the different methods they have used to accomplish this task. Some groups are asked to go back and reexamine their results. When the class is convinced that all of the results are reasonably accurate, they consider how the surface area of the leaf might be related to the growth of the tree and its needs for carbon dioxide, sunshine, and water.
- Each group of students is given a mixing bowl and asked to find its volume. One group

decides to fill the bowl with centimeter cubes, packing them as tightly as they can and then to add a little. Another group decides to turn the bowl upside down and try to build the same shape next to it by making layers of centimeter cubes. Still another group decides to fill the hollow 1000-centimeter cube with water and empty it into the bowl as many times as they can to fill it; they find that doing this three times almost fills the bowl and add 24 centimeter cubes to bring the water level up to the top of the bowl.

11. Express mathematically and explain the impact of the change of an object's linear dimensions on its surface area and volume.

- While learning about area, the students became curious about how many square inches there are in a square foot. Some students thought it would be 12, while others thought it might be more. They explore this question using square-inch tiles to make a square that is one foot on each side. They decide that there are 144 square inches in a square foot; they make the connection with multiplication, noticing that 144 is 12 x 12 and that there are 12 inches in a foot. They realize that the square numbers have that name because they are the areas of squares whose sides are the whole numbers.
- Having measured the length, width, and height of the classroom in feet, the students now must find how many cubic yards of air there are. Some of the students convert their measurements to yards and then multiply to find the volume. Others multiply first, but find that dividing by 3 does not give a reasonable answer. They make a model using cubes that shows that there are 27 cubic feet in a cubic yard and divide their answer by 27, getting the same result as the other students.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards.*

Standard 15 — Conceptual Building Blocks of Calculus— Grade 7-8

Overview

In grades 7 and 8, students begin to develop a more detailed and formal notion of the concepts of approximation, rates of change for various quantities, infinitely repeating processes, and limits. Activities should continue to emerge from concrete, physical situations, often involving the collection of data.

Students in grades 7 and 8 continue to develop their understanding of **linear growth, exponential growth, infinity,** and **change over time**. By collecting data in many different situations, they come to see the commonalities and differences in these types of situations. They should recognize that, in linear situations, the rate of change is constant and the graph is a straight line, as in plotting distance vs. time at constant speeds or plotting the height of a candle vs. time as it burns. In situations involving exponential growth, the graph is not a straight line and the rate of change increases or decreases over time. For example, in a situation in which a population of fish triples every year, the number of fish added each year is more than it was in the previous year. Students should also have some experience with graphs with holes or jumps (discontinuities) in them. For example, students may look at how the price of mailing a letter has changed over the last hundred years, first making a table and then generating a graph. They should recognize that plotting points and then connecting them with straight lines is inappropriate, since the cost of mailing a letter stayed constant over a period of several years and then abruptly increased. They should be aware of other examples of “step functions” whose graphs look like a sequence of steps.

Students in these grades should approximate irrational numbers, such as square roots, by using decimals; they should recognize the size of the error when they use these approximations. Students should take care to note that π is a nonterminating, nonrepeating decimal; it is not exactly equal to 22/7 or 3.14, but lies between these values. These approximations are fairly close to the actual value of π and can usually be used for computational purposes. Students may also consider sequences involving rational numbers such as 1/2, 2/3, 3/4, 4/5, ... They should recognize that this sequence goes on forever, getting very close to a limit of one. Students should also consider sequences in the context of learning about fractals. (See Standard 7 or 14.)

Seventh and eighth graders continue to benefit from activities that physically model the process of approximating **measurement** results with increasing accuracy. Students should develop a clearer understanding of the concept of significant digits as they begin to use scientific notation. They should be able to use these ideas as they develop and apply the formulas for finding the areas of such figures as parallelograms and trapezoids. Students should understand, for example, that if they are measuring the height and diameter of a cylinder in order to find its volume, then some error is introduced from each of these measurements. If they measure the height as 12.2 cm and the diameter as 8.3 cm, then they will get a volume of $\pi(8.3/2)^2(12.2)$, which their calculator may compute as being $660.09417\ cm^3$. They need to understand that this answer should be rounded off to $660.09\ cm^3$ (five significant digits). They also should understand that the true volume might be as low as $\pi(8.25/2)^2(12.15) \approx 649.49\ cm^3$ or almost as high as $\pi(8.35/2)^2(12.25) \approx 670.81\ cm^3$.

Students in these grades should continue to build a repertoire of strategies for finding the surface area and volume of irregularly shaped objects. For example, they might find volume not only by approximating irregular shapes with familiar solids but also by submerging objects in water and finding the amount of water displaced by the object. They might find surface areas by first laying out patterns of the objects called “nets”;

for example, the net of a cube consists of six squares connected in the shape of a cross — when creased along the edges of the squares, this “net” can be folded to form a cube. Then they would place a grid on the net and count the small squares, noting that the finer the grid the more accurate the estimate of the area.

Explorations developing the conceptual underpinnings of calculus in grades 7 and 8 should continue to take advantage of students’ intrinsic interest in infinite, iterative patterns. They should also build connections between number sense, estimation, measurement, patterns, data analysis, and algebra. More information about activities related to these areas can be found in the chapters discussing those standards.

Standard 15 — Conceptual Building Blocks of Calculus— Grades 7-8

Indicators and Activities

The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 7 and 8.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 7-8 will be such that all students:

4. Recognize and express the difference between linear and exponential growth.

- Students measure the height of water in a beaker at five second intervals as it is being filled, being careful to leave the faucet on so that the water runs at a constant rate. They make a table of their results and generate a graph. They note that this is a linear function.
- Students investigate patterns of exponential growth with the calculator, such as compound interest or bacterial growth. They make a table showing how much money is in a savings account after one quarter, two quarters and so on for ten years, if $1000 is deposited at 5% interest and there are no further deposits or withdrawals. They represent their findings graphically, noting that this is not a linear relationship, although in the case of simple interest, where the interest does not earn interest, the graph is linear.
- Students obtain a table showing the depreciated value of a car over time. They graph the data in the table and observe that it is not a straight line. The value of the car exhibits "exponential decay."
- Students compare different pay scales, deciding which is a better deal. *For example, is it better to be paid a salary of $250 per week or to be paid $6 per hour?* They realize that the answer to this question depends on the number of hours worked, so they create a table comparing the pay for different numbers of hours worked. They make a graph and decide at what point the hourly rate becomes a better deal.
- Students predict how many times they will be able to fold a piece of paper in half. Then they fold a paper in half repeatedly, recording the number of sections formed each time in a table. Students find that the number of folds physically possible is surprisingly small (about 7). The students try different kinds of paper: tissue paper, foil, etc. They describe in writing any patterns they discover and try to find a rule for the number of sections after 10, 20, or *n* folds. They also graph the data on a rectangular coordinate plane using integral values. They extend this problem to a new situation by finding the number of ancestors each person had perhaps ten generations ago and also to the situation of telling a secret to 2 people who each tell two people, etc.

5. Develop an understanding of infinite sequences that arise in natural situations.

- Students discuss how the Fibonacci sequence (1, 1, 2, 3, 5, 8, 13, ...) is related to the following problem: begin with two rabbits (one male and one female), each adult pair of rabbits produces two babies (one male and one female) each month, the babies themselves become adults (and start having their own babies) after one month, and none of the rabbits ever die. The students decide that the Fibonacci sequence shows how many pairs of rabbits there are each month. The students explore other patterns in this sequence, noting that each term is the sum of the two preceding terms.

- Students look for infinite sequences in Pascal's triangle. Starting at the top 1 and moving diagonally to the left, there is a constant infinite sequence 1, 1, 1, 1, ... Starting at the next 1 and moving diagonally to the left there is the sequence 1, 2, 3, 4, 5, ... of whole numbers. Starting at the next 1 and moving diagonally to the left, there is the sequence 1, 3, 6, 10, 15, ... of triangular numbers, which records the solutions to all handshake problems. Also the sum of the numbers in each row yield the exponential sequence
1, 2, 4, 8, 16, ...

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

6. Investigate, represent, and use non-terminating decimals.

- Students investigate using simple equations to iterate patterns. For example, they use the equation $y = x + 1$ and start with any x-value, say 0. The resulting y-value is 1. Using this as the new x yields a 2 for y. Using this as the next x gives a 3, and so on, resulting in the sequence 1, 2, 3, 4, ... Then students use a slightly different equation, $y = .1x + .6$, starting with an x-value of .6 and finding the resulting y-value. Repeating this process yields the sequence of y-values *.6, .66, .666, .6666, ...*, which approximates the decimal value of 2/3.

- Students explore the question of which fractions have terminating decimal equivalents and which have repeating decimal equivalents. They discover that the only fractions in lowest terms which correspond to terminating decimals are those whose denominators have only 2 and 5 as prime factors.

- Students explore the question of which fractions have decimal equivalents where one digit repeats and learn that these are the fractions 1/9, 2/9, 3/9, ... They generalize this to find the fractions whose decimal equivalents have two digits repeating like .171717 ... .

7. Represent, analyze, and predict relations between quantities, especially quantities changing over time.

- Students describe what happens when a ball is tossed into the air, experimenting with a ball as needed. They make a graph that shows the height of the ball at different times and discuss what makes the ball come back down. They also consider the speed of the ball: *when is it going fastest? slowest?* With some help from the teacher, they make a graph showing the speed of the ball over time.

- Students use probes and graphing calculators or computers to collect data involving two variables for several different science experiments (such as measuring the time and distance that a toy car rolls down an inclined plane or measuring the brightness of a light bulb as the distance from the light bulb increases or measuring the temperature of a beaker of water when ice cubes are added). They look at the data that has been collected in tabular form and as a graph on a coordinate grid. They classify the graphs as straight or curved lines and as increasing (direct variation), decreasing (inverse variation), or mixed. For those graphs that are straight lines, the students try to match the graph by entering and graphing a suitable equation.

- Students measure the temperature of boiling water as it cools in a cup. They make a table showing the temperature at five-minute intervals for an hour. Then they graph the results and make observations about the shape of the graph, such as "the temperature went down the most in the first few minutes," "it cooled more slowly after more time had passed," or "it's not a linear relationship." The students also predict what the graph would look like if they continued to collect data for another twelve hours.

- Students make Ferris wheel models from paper plates (with notches cut to represent the cars). They use the models to make a table showing the height above the ground (desk) of a person on a Ferris wheel at specified time intervals (time needed for next chair to move to loading position). After collecting data through two or three complete turns of the wheel, they make a graph of time versus height. In their math notebooks, they respond to questions about their graphs: *Why doesn't the graph start at zero? What is the maximum height? Why does the shape of the graph repeat?* The students learn that this graph represents a periodic function.

- Students compare two ways of cooling a glass of soda, adding lots of ice at the beginning or adding one cube at a time at one minute intervals. Each student first makes a prediction about which cools the soda faster, and the class summarizes the predictions. Then the teacher collects the data, using probes and graphing calculators or computers and displays the results in table and graph form on the overhead. The students compare the graphs and write their conclusions in their math notebooks. They discuss the reasons for any difference between these two methods with their science teacher.

- Students compute the average speed of a toy car as it travels down a ramp by dividing the length of the ramp by the time the car takes to travel the ramp. They try different angles for the ramp, recording their results. They make a graph of average speed vs. angle and discuss whether this graph is linear.

- Students make a graph that shows the minimum wage from the time it was first instituted until the present day. Some of the students begin by simply plotting points and connecting them but soon realize that the minimum wage was constant for a time and then abruptly jumped up. They decide that parts of this graph are like horizontal lines. They look for other examples of "step functions."

8. Approximate quantities with increasing degrees of accuracy.

- Students measure the speed of cars using different strategies and instruments and compare the accuracy of each. For example, they first determine the speed of a car by using a stopwatch to find out how long it takes to travel a specific distance. They note that the speed of the car actually changes over the time interval, however. They decide that they can get a better idea of how fast the car is moving at a specific time by shortening the distance. They collect data for shorter and shorter distances. Finally, they ask a police officer to bring a radar gun to their class to help them collect data about the speed of the cars going past the school.

- Students find the area of a "blob" using a square grid. First, they count the number of squares that fit entirely within the blob (no parts hanging outside). They say that this is the least that the area could be. Then they count the number of squares that have any part of the blob in them. They say that this is the most that the area could be. They note that the actual area is somewhere between these two numbers. Finally, the students put together parts of squares to try to get a more accurate estimate of the area of the blob.

9. Understand and use the concept of significant digits.

- Students measure the radius of a circle in centimeters and find its area. Then they measure its radius in millimeters and find the area. They note the difference between these two results and discuss the reasons for such a difference. Some of the students think that, since the original measurements were correct to the nearest centimeter, then the result would be correct to the nearest square centimeter, while using the second measurements would give a value for the area which is correct to the nearest square millimeter. However, after experimenting with circles of different sizes, they find that if the radius is measured to the nearest centimeter.
- Students explore the different answers that they get by using different values for π when finding the area of a circle. They discuss why these answers vary and how to decide what value to use.
- Students estimate the amount of wallpaper, paint, or carpet needed for a room, recognizing that measurements that are accurate to several decimal places are unnecessary for this purpose.

10. Develop informal ways of approximating the surface area and volume of familiar objects, and discuss whether the approximations make sense.

- In conjunction with a science project, students need to find the surface area of their bodies. Some of the students decide to approximate their bodies with geometric solids; for example, their head is approximately a sphere, and their neck, arms, and legs are approximately cylinders. They then take the needed measurements and compute the surface areas of the relevant solids. Other students decide to use newspaper to wrap their bodies and then measure the dimensions of the sheets of newspaper used.
- Students estimate the volume of air in a balloon as a way of looking at lung capacity. Some of the students decide that the balloon is approximately the shape of a cylinder, measure its length and diameter, and compute the volume. Other students think the balloon is shaped more like a cylinder with cones at the ends; they measure the diameter of the balloon at its widest part, the length of the cylinder part, and the height of each cone and then compute the volume of each shape. Some other students decide that they would like to check their work another way; they place a large graduated cylinder in the sink, and fill it with water. They submerge the balloon, and read off how much water is left after the balloon is taken out. Since they know that 1 ml of water is 1 cm^3, they know that the volume of the water that was displaced is the same as that of the balloon.
- Students develop different strategies for finding the volume of water in a puddle.

11. Express mathematically and explain the impact of the change of an object's linear dimensions on its surface area and volume.

- Students analyze cardboard milk containers to determine how the dimensions of the container affect the volume of milk contained in the carton and how the amount of cardboard used varies. In addition to measuring actual cartons, students make their own cartons of different sizes by varying the length, width, and height one at a time. They write up their results and share them with the class.
- Placing a number of identical cereal boxes next to or on top of one another, students learn that doubling one of an object's length, width, or height doubles its volume, that doubling two of these dimensions increases the volume by a factor of 4, and that doubling all three dimensions increases the volume by a factor of 8.
- Students sketch a 3-dimensional object such as a box or a cylindrical trash can. They then make a sketch twice as large in all dimensions. *How much larger is the volume of the larger object? How much larger would it be if the dimensions all increased by a factor of 3?* Square grid paper might be helpful for this exercise.

On-Line Resources

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The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards.*

Standard 15 —Conceptual Building Blocks of Calculus — Grades 9-12

Overview

This standard does **not** advocate the formal study of calculus in high school for all students or even for all college-intending students. Rather, it calls for providing opportunities for all students to informally investigate the central ideas of calculus: limit, the rate of change, the area under a curve, and the slope of a tangent line. Considering these concepts will contribute to a deeper understanding of the function concept and its usefulness in representing and answering questions about real-world situations.

Instruction should be highly exploratory, based on numerical and geometric experiences that capitalize on both calculator and computer technology. Activities should be aimed at providing students with an understanding of the underlying concepts of calculus rather that at developing manipulative techniques.

The development of calculus is one of the great intellectual achievements in history, especially with respect to its use in physics. Calculus is also increasingly being used in the social and biological sciences and in business. As students explore this area, they should develop an awareness of and appreciation for the historical origins and cultural contributions of calculus.

Students' earlier study of patterns is extended in high school to the study of finite and **infinite processes**. Students continue to look at **linear growth** patterns as they develop procedures for finding the sums of arithmetic series (e.g., the sum of the numbers from 1 to 100). They may consider this sum in many different ways, building different types of models. Some students may look at $1 + 2 + 3 + ... + 100$ geometrically by putting together two "staircases" to form a rectangle that is 100 by 101. Other students may look at the sum arithmetically by adding *1 + 2 + 3 + ... + 100 to 100 + 99 + 98 + ... + 1* and getting 100 pairs of numbers that add up to 101. Still others may look at the sum by finding the limit of the sequence of partial sums. Students also look at **exponential growth** as they develop procedures for finding the sum of finite and infinite geometric series (e.g., $2 + 4 + 8 + 16 + 32$ or $6 + 3 + 3/2 + ...$ or finding the total distance traveled by a bouncing ball). Students' work with patterns and infinity also includes elaborating on the intuitive notion of limit that has been addressed in the earlier grades.

High school students further develop their understanding of **change over time** through informal activities that focus on the understanding of interrelationships. Students should collect data, generate graphs, and analyze the results for real-world situations that can be described by linear, quadratic, trigonometric, and exponential models. Some of the types of situations that should be analyzed include motion, epidemics, carbon dating, pendulums, and biological and economic growth. They should use Calculator Based Labs (CBLs) in conjunction with graphing calculators to gather and analyze data. Students should recognize the equations of the basic models ($y = mx + b$, $y = ax^2 + bx + c$, $y=\sin x$, and $y = 2^x$) and be able to relate geometric transformations to the equations of these models. Students should develop a thorough understanding of the idea of slope; for example, they need to be able compare the steepness of two graphs at various points on the graph. They also need to be able explain what the slope means in terms of the real-world situation described by a graph. *For example, what information does the slope give for a graph of the levels of medicine in the bloodstream over time?* Students also extend their understanding of the behavior of functions to include the concept of the continuity of a function, considering features such as removable discontinuities (holes or jumps), asymptotes, and corners.

Students in high school apply their understanding of approximation techniques not only with respect to numbers in the context of using initial portions of nonrepeating, nonterminating decimals but also with respect to **measurement** situations. Students further develop their understanding of significant digits and the arithmetic of approximate values. They also use repeated approximations to find the areas of irregular figures, including experimenting with situations in which they need to find the area under a curve.

Looking at the conceptual underpinnings of calculus provides an opportunity for high school students to pull together their experiences with data analysis, patterns, algebra, measurement, number sense, and numerical operations. It also provides the opportunity to apply technology to real-world situations and to gain experience with mathematics as a dynamic human endeavor.

Standard 15 — Conceptual Underpinnings of Calculus — Grades 9-12

Indicators and Activities

The cumulative progress indicators for grade 12 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 9, 10, 11, and 12.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 9-12 will be such that all students:

12. Develop and use models based on sequences and series.

- Students work in groups to collect data about a bouncing ball. They first decide how to measure the height of each bounce and then record their data in a table. They notice the pattern of the heights and make two graphs, one of height vs. bounce and the other of total distance traveled vs. bounce. They note that the distance traveled involves adding together the heights of each of the bounces, and so is represented by a series. They describe the general behavior of each graph and have their graphing calculators compute various regression lines. In their report, they describe what they did, their results, and why they think that the type of function they used to describe each graph is reasonable.
- Students use M&Ms to model exponential decay. They spill a package of M&Ms on a paper plate and remove those with the M showing, and record the number of M&Ms removed. They put the remaining M&Ms in a cup, shake, and repeat the process until all of the M&Ms are gone. They plot the trial number versus the number of M&Ms removed and note that the graph represents an exponential function. Some of the students try out different exponential functions until they find one that they think fits the data pretty well.

13. Develop and apply procedures for finding the sum of finite arithmetic series and finite and infinite geometric series.

- Students investigate a situation in which a contractor is fined $400 if he is one day late completing a project, $475 more if he is two days late, $550 more if he is three days late, and so on. They want to find out how much he will lose if he is two weeks late finishing the job. They recognize that this is an arithmetic series where the first term is $400 and each term is obtained from the preceding one by adding $75. They draw upon several techniques they have learned to add up the terms of this series. One method that they have discussed involves reversing the order of the terms of the series and adding the two series. Some of the students thus solve the problem by writing the fourteen terms of the series and underneath writing the same fourteen terms backwards, a technique sometimes called Gauss' method because, according to legend, he discovered it as a child while walking to the back of his class to perform his punishment of adding together the first hundred numbers. They obtain the following format:

$$\begin{array}{l} \;\;400 + \;\;475 + \;\;550 + \ldots + 1300 + 1375 \\ 1375 + 1300 + 1225 + \ldots + \;\;475 + \;\;400 \\ \hline 1775 + 1775 + 1775 + \ldots + 1775 + 1775 \end{array}$$

They recognize that they have 14 pairs of numbers, each of which adds up to 1775. This gives them a total of \$24,850 which they divide in half (since they added together both sequences) to find the answer, \$12,425. Another group decides that they can separate out the 14 charges of \$400, for a total of *14 x 400 = \$5600,* and then deal with the remainders \$0 + 75 + 150 + 225 + ... + 975, or \$75(0 + 1 + 2 + 3 + ... + 13); this series they recognize as (13x14)/2, so the total fine is \$5600 + \$75x91 or \$5600 + \$6825, for a grand total of \$12,425. Still another group of students uses a formula for the sum of a finite arithmetic series.

- Students are asked to find a method similar to Gauss' method to find the sum of the series *9 + 3 + 1 + 1/3 + 1/9 + 1/27*. The students notice that this series is not an arithmetic series since different amounts have to be added in order to get the next term. They discover, however, that each term is 1/3 of the previous term, and they write down 1/3 of the series and arrive at:

$$\begin{array}{l} 9 + 3 + 1 + 1/3 + 1/9 \\ \quad\;\; 3 + 1 + 1/3 + 1/9 + 1/27 \\ \hline \end{array}$$

  They subtract to get 9 - 1/27 or 242/27 . Since they subtracted 1/3 of the series from itself, this total is 2/3 of the sum of the series, so the sum is 3x242/2x27 or 121/9. The teacher uses this technique to motivate the standard formula for the sum of a finite geometric series, where a is the first term of the series and r is the common multiple:

$$S_n = a\,(1 \quad r^n)/(1 \quad r).$$

- After investigating how to find the sum of a finite geometric sequence, students begin looking at infinite geometric sequences. They realize that the same technique they used for the finite geometric series works for the infinite one as well. Thus for example, if we added the first 100 terms of the series by the method above, the sum would be $9 - 1/3^{97}$, which is very close to 9. Since this sum is again 2/3 of the sum of the original series, the actual total is 27/2. For those students who are likely to use a formula, the teacher generalizes this discussion and tells them that the sum gets closer to a/(1 r) as the number of terms expand. They confirm this conclusion by checking out the partial sums of some sequences.

14. Develop an informal notion of limit.

- After a class discussion of the repeating decimal .9999 ... , the students are asked to write in their journals an "explanation to the skeptic" on why .9999 ... is equal to 1. Among their explanations: There is no room between .9999 ... and 1; .9999 ... is 3 times .3333 ... which everyone agrees is 1/3 ; if you take 10 times .9999 ... and subtract .9999 ... , you get 9 and 9 times .9999 ... — so 1 must be .9999 ... ; if you sum a geometric series whose first time is .9 and whose common multiple is .1 you get a/(1-r) which amounts to .9/(1 .9), or 9. Given all these convincing reasons, the class decides that the limit of the sequence is 1.

- Students consider the sequence 1/2, 1/4, 1/8, ... in different contexts. First, they look at it as representing a situation in which someone eats half of a pizza, then half of what is left, then half of what is left, etc. They decide that, while theoretically there will always be some of pizza left, in the end it would be all gone. However, in practice, by the end of ten stages or so the entire pizza would in effect have disappeared. Similarly, if a sheet of paper is repeatedly torn in half, then in theory some part is always left; however, in practice, after about ten tearings the paper will have disappeared.

15. Use linear, quadratic, trigonometric, and exponential models to explain growth and change in the natural world.

- Students use a graphing calculator, together with a light probe, to examine the relationship between brightness of a light and distance from it. They do this by collecting data with the probe on the brightness of a light bulb at increasing distances and then analyzing the graph generated on the calculator to see what kind of graph it is. They use other CBL probes to investigate the kinds of functions used to model a variety of real-world situations.

- Students learn about the Richter Scale for measuring earthquakes, focusing on its relation to logarithmic and exponential functions, and why this kind of scale is used.

- Students use recursive definitions of functions in both geometry and algebra. For example, they define n! recursively as $n! = n\,(n-1)!$ They use recursion to generate fractals in studying geometry. They may use patterns such as spirolaterals, the Koch snowflake, the *monkey's tree curve*, the *chaos game*, or the Sierpinski triangle. They may use Logo or other computer programs to iterate patterns, or they may use the graphing calculator. In studying algebra, students consider the equation $y = .1x + .6$, start with an x-value of .6, and find the resulting y-value. Using this y-value as the new x-value, they then calculate its corresponding y-value, and so on. (The resulting values are .6, .66, .666, .6666, etc. — an approximation to the decimal value of 2/3!) Students investigate using other starting values for the same function; the results are surprising! They use other equations and repeat the procedure. They graph the results and investigate the behavior of the resulting functions, using a calculator to reduce the computational burden.

- Students work through the *Breaking the Mold* lesson described in the Introduction to this *Framework*. They grow mold and collect data on the area of a pie plate covered by the mold. They make a graph showing the percent of increase in the area vs. the days. The students graph their data and find an equation that fits the data to their satisfaction.

16. Recognize fundamental mathematical models (such as polynomial, exponential, and trigonometric functions) and apply basic translations, reflections, and dilations to their graphs.

- Students work in groups to investigate what size square to cut from each corner of a rectangular piece of cardboard in order to make the largest possible open-top box. They make models, record the size of the square and the volume for each model, and plot the points on a graph. They note that the relationship seems to be a polynomial function and make a conjecture about the maximum volume, based on the graph. The students also generate a symbolic expression describing this situation and check to see if it matches their data by using a graphing calculator.

- Students look at the effects of changing the coefficients of a trigonometric equation on the graph. *For example, how is the graph of* $y = 4 \sin x$ *different from that of* $y = \sin x$*? How is* $y = .2 \sin x$ *different from* $y = \sin x$*? How are* $y = \sin x + 4$, $y = \sin x - 4$, $y = \sin (x - 4)$, *and* $y = \sin (x + 4)$ *each different from* $y = \sin x$*?* Students use graphing calculators to look at the graphs and summarize their conjectures in writing.

- Students study the behavior of functions of the form $y = ax^n$. They investigate the effect of "a" on the curve and the characteristics of the graph when n is even or odd. They use the graphing calculator to assist them and write a sentence summarizing their discoveries.

- Students begin with the graph of $y = 2^x$. They shift the graph up one unit and try to find the equation of the resulting curve. They shift the original graph one unit to the right and try to find the equation of that curve. They reflect the original graph across the x-axis and try to find the equation of that curve. Finally, they reflect the original graph across the y-axis and try to find the equation of the resulting curve. They describe what they have learned in their journals.

17. Develop the concept of the slope of a curve, apply slopes to measure the steepness of curves, interpret the meaning of the slope of a curve for a given graph, and use the slope to discuss the information contained in the graph.

- Students collect data about the height of a ball that is thrown in the air and make a scatterplot of their data. They note that the points lie on a quadratic function and use their graphing calculators to find the curve of best fit. Then they make some conjectures about the speed at which the ball is traveling. They think that the ball is slowing down as it rises, stopping at the maximum point, and speeding up again as it falls.
- Students take on the role of "forensic mathematicians," trying to determine how tall a person would be whose femur is 17 inches long. They measure their own femurs and their heights, entering this data into a graphing calculator or computer and creating a scatterplot. They note that the data are approximately linear, so they find the y-intercept and slope from the graph and generate an equation that they think will fit the data. They graph their equation and check its fit. They also use the built-in linear regression procedure to find the line of best fit and compare that equation to the one they generated. (An instructional unit addressing this activity can be found in the Keys to Success in the Classroom chapter of this *Framework.*)
- Students plot the data from a table that gives the amount of alcohol in the bloodstream at various intervals of time after a person drinks two glasses of beer. Different groups use different techniques to generate an equation for the graph; after some discussion, the class decides which equation they think is best. The students consider the following questions: *What information does the slope give for this situation? Would that be important to know? Why or why not?*
- Students investigate the effect of changing the radius of a circle upon its circumference by measuring the radius and the circumference of circular objects. They graph the values they have generated, notice that it is close to a straight line, and use the slope to develop an equation that describes that relationship. Then they discuss the meaning of the slope in this situation.

18. Develop an understanding of the concept of continuity of a function.

- Students work through the *On the Boardwalk* lesson found in the Introduction to this *Framework*. A quarter is thrown onto a grid made up of squares, and you win if the quarter does not touch a line. A grid is drawn on the floor using masking tape, and a circular paper plate is thrown onto the grid several hundred times to simulate the game. The activity is repeated several times, varying each time the size of the squares in the grid. The students collect data and make a graph of their results (size of squares vs. number of wins out of 100 tosses). The graph looks like a straight line, suggesting that as the size of the squares increases without bound, so does the percentage of "hits". But, of course, the percentage of hits cannot exceed 100%, so the line is actually curved, with an asymptote at y=100.

- The school store sells pencils for 15¢ each, but it has some bulk pricing available if you need more pencils. Ten pencils sell for $1, and twenty-five pencils sell for $2. The students make a table showing the cost of different numbers of pencils and then generate a graph of number of pencils vs. cost. The students note that the graph has discontinuities at ten and twenty-five, since these are the jump points for pricing. They also note that if you need at least seven pencils, it is better to buy the package of ten and if you need 17 or more, you should get the package of 25.

- Students make a table, plot a graph (number of people vs. cost), and look for a function to describe a situation in which the Student Council is sponsoring a Valentine's Day dance and must pay $300 to the band, no matter how many people come. They also must pay $4 per person for refreshments, with a minimum of 50 people. The students note that the cost will be $500 for anywhere from 0-50 people and then increase at a rate of $4 per person. They decide that this is a function with a corner and needs to be defined in pieces:

  $f(x) = 500$ for x 50
  $f(x) = 500 + 4x$ for x >50

19. Understand and apply approximation techniques to situations involving initial portions of infinite decimals and measurement.

- Students investigate finding the area under the curve $y = x^2 + 1$ between -1 and 1. They approximate the area geometrically by dividing it into rectangles 0.5 units wide. They find the height of each rectangle that fits under the curve and use it to find the areas. Then they find the height of each rectangle that contains the curve and use these measurements to find the areas. They realize that this gives them a range of values for the area under the curve. They refine this approximation by using narrower rectangles, such as 0.1.

- After some experience with collecting data about balls thrown into the air, students are given a table of data about a model rocket and its height at different times. They plot the data, find an equation that fits the data, and use the trace functions on their graphing calculators to find the maximum height.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/
The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards.*